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Mathematical modelling of a two-phase radial flow in a hydrocarbon reservoir and analytical solution for nonlinear diffusion problem

Amirabbas Mohammadi[1], Abdullah Shidfar[1], Mostafa Mohammadi[2], Amir H. Mohammadi[3]*

[1] *School of Mathematics, Iran University of Science and Technology, Tehran, Iran*
[2] *Department of Mathematics, Shahid Beheshti University, Tehran, Iran*
[3] *Discipline of Chemical Engineering, School of Engineering, University of KwaZulu-Natal, Howard College Campus, King George V Avenue, Durban 4041, South Africa*

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Abstract

In this work, the radial flow of a two-phase hydrocarbon fluid in a petroleum reservoir is first modeled. Using the mass balance equation and Darcy's law as the governing equations, a nonlinear radial diffusion equation is obtained. We then assume that throughout the reservoir, $\partial P/\partial r$ is small and $\partial P/\partial r \neq 0$ also the viscosity (μ) is independent of pressure. The recent hypotheses are true about the radial flow. Accordingly, the resulting nonlinear equation is converted to a linear radial diffusion equation which is a one-dimensional equation (ODE) and it is noticeable to achieve the pressure distribution in the reservoir and adjacent to the well. Descriptions of reservoir processes have motivated a large volume of work on ODEs. Analytical and numerical methods have provided solutions to the problems satisfying a fairly wide range of conditions. However, analytical methods continue to be highly valued for the inherent simplicity, their capacity to convey qualitative information about the physical problem, and as a verification for numerical models. In this work, after obtaining the linear radial diffusion equation, regarding the transient state ($\partial P/\partial r = f(r,t)$ and $\partial P/\partial r \neq 0$) for the fluid flow, we apply the related boundary and initial conditions. Afterward, we apply the analytical methods for the resulting model to obtain the pressure distribution. To achieve this goal, we separately apply the separation of variables method (based on Bessel equations) and the method of Laplace transform to solve the problem. In the end of the second method, as for the resulted fraction any inverse value is not found in most of tables, so we apply Heaviside's theorem.

***Keywords**: Radial flow; Darcy's law; Transient state; Bessel equation; Laplace transform.*

1. Introduction

Generally, after digging a well in a petroleum reservoir, the difference in pressure between the reservoir and the wellbore causes to the flow of fluid into the wellbore in three states: transient ($\frac{\partial P}{\partial t} \neq 0$), semi-steady ($\frac{\partial P}{\partial t} = \text{constant}$) and steady ($\frac{\partial P}{\partial t} = 0$) [1].

When digging a well in the middle of the reservoir, the hydrocarbon fluid flows from the surrounding area to the wellbore, then a radial flow occurs (Fig. 1). When numerous wells are digged at a short distance from each other, the flow lines are parallel and a linear flow is created. The spherical flow occurs when the reservoir has a nearly hemispherical shape [2]. In this study, we assumed that the desired fluid has a radial flow in a hydrocarbon reservoir.

The pressure distribution at any time (t) and at any point (r) in a reservoir producing a hydrocarbon fluid with radial flow is an applied function that is used daily by reservoir engineers in order to predict pressure drop and provide methods of stabilizing or increasing the pressure. Therefore, it is significant to provide a suitable model for determining the distribution of this important parameter in a reservoir.

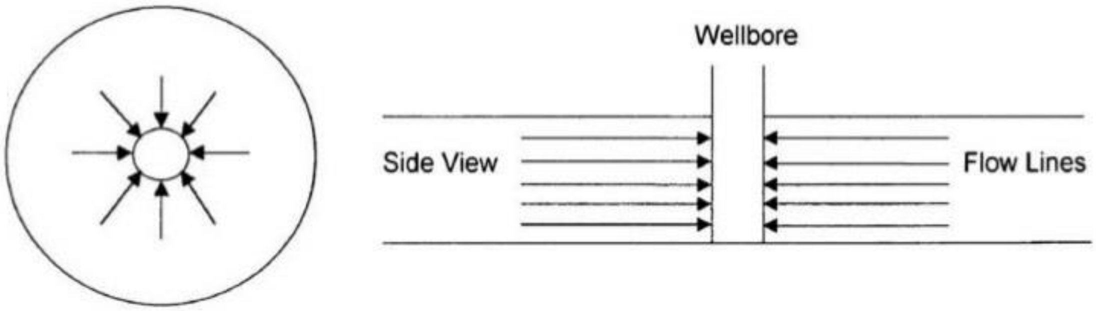

Figure 1. Radial flow into a wellbore

In recent years, various studies have been carried out to calculate the distribution of pressure in the reservoir. In this regard, the use of the diffusion equation in calculations has been of interest to many researchers. It seems that provision of analytical models (if applicable) with mathematical structures can play a decisive role in proving the validity and reliability of numerical models in addition to problem solving. The analytical method of diffusion equation, which is superior in calculation to the existing analytical methods such as the integral transformation method of Fourier or Laplace and/or the variable separation method, was thus established in the parabolic space [3].

In this work, in order to achieve the pressure distribution throughout the reservoir, the problem is only studied in the transient state which is undoubtedly difficult to analyze. The semi-steady and steady states (due to the simplicity of the problem) will not be studied in this work.

2. Mathematical formulation

We consider the following assumptions:

A. In a hydrocarbon reservoir, since deposition of some materials such as asphaltene, causes to changing the porosity in terms of time, so we ignore the deposition and assume that physical properties of the reservoir including porosity and permeability are constant. In general, we assume that the reservoir is homogeneous.
B. Height of drilling well is equal to the reservoir thickness, in this case, assumption of radial flow for the fluid is correct.
C. In a two-phase reservoir, the desired fluid is oil or gas and water is immobile.

The mass balance equation is considered as "rate of mass accumulation = rate of input mass –rate of output mass" [4].

According to Fig. 2 and defined parameters earlier, the mass balance equation can be written as follow:

$$q\rho|_{r+dr} - q\rho|_{r} = (2\pi rh\varphi dr)\frac{d\rho}{dt} \quad \text{or} \quad \frac{q\rho|_{r+dr} - q\rho|_{r}}{dr} = (2\pi rh\varphi)\frac{d\rho}{dt} \tag{1}$$

where: $2\pi rh\varphi dr$ is the volume of vacant space in the cylindrical layer in which the fluid is placed. Equation (1) can be written as below:

$$\frac{\partial}{\partial r}(q\rho) = (2\pi rh\varphi)\frac{d\rho}{dt} \tag{2}$$

On the other hand, for a radial flow in horizontal mode, Darcy's law is [5-6]:

$$q = \frac{2\pi rkh}{\mu}\frac{\partial\rho}{\partial r} \tag{3}$$

where: $2\pi rh$ is the surface which is perpendicular to the direction of flow. Using equations (2) and (3) we can obtain:

$$\frac{\partial}{\partial r}\left(\frac{2\pi rkh}{\mu}\rho\frac{\partial\rho}{\partial r}\right) = (2\pi rh\varphi)\frac{d\rho}{dt} \tag{4}$$

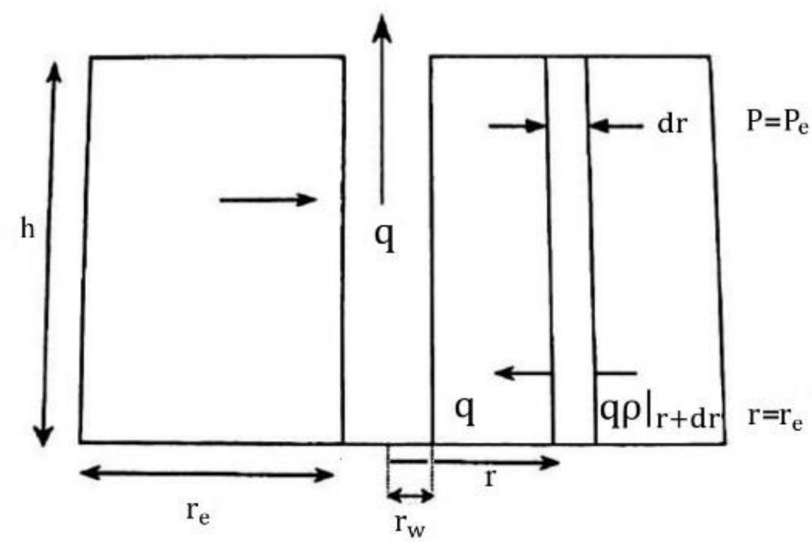

Figure 2. A cylindrical part of a hydrocarbon reservoir with length h (thickness of the reservoir) and thickness dr and inner radius r

The above equation can be written as follow:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k\rho}{\mu}r\frac{\partial\rho}{\partial r}\right) = \varphi\frac{d\rho}{dt} \tag{5}$$

Considering the compressibility of fluid is defined as follow:

$$c = -\frac{1}{v}\frac{\partial v}{\partial P} \tag{6}$$

Hence, according to definition of mass density ($\rho = \frac{m}{v}$), we can have:

$$c = -\frac{\rho}{m}\frac{\partial(\frac{m}{\rho})}{\partial P} = \frac{1}{\rho}\frac{\partial\rho}{\partial P} \tag{7}$$

By differentiating of both sides of equation (7) in terms of time, we can have:

$$c\rho\frac{\partial P}{\partial t} = \frac{\partial\rho}{\partial t} \tag{8}$$

By substituting equation (8) in (5):

$$\frac{1}{r}\frac{\partial}{\partial r}(\frac{k\rho}{\mu}r\frac{\partial\rho}{\partial r}) = \varphi c\rho\frac{\partial P}{\partial t} \tag{9}$$

Equation (9) is called the radial diffusion equation in the reservoir which is nonlinear because the pressure indirectly affects the density, compressibility and viscosity. In general, equation (9) does not have any simple solution but in particular conditions can be linearized and can be solved. After differentiation, equation (9) can be extended as follow:

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(\frac{k}{\mu})\rho r\frac{\partial\rho}{\partial r}+(\frac{k}{r})\frac{\partial\rho}{\partial r}r\frac{\partial\rho}{\partial r}+(\frac{k\rho}{\mu})\frac{\partial\rho}{\partial r}+(\frac{k\rho}{\mu})r\frac{\partial^2\rho}{\partial r^2}\right\}=\varphi c\rho\frac{\partial P}{\partial t} \tag{10}$$

On the other hand, equation (7) can be written as follow:

$$\frac{\partial P}{\partial r}=\frac{1}{c\rho}\frac{\partial\rho}{\partial r} \tag{11}$$

In a two-phase reservoir of oil or gas and water (water is considered immobile), the following assumptions are true about the radial flow:

A. Viscosity (μ) is independent of pressure.

B. $\frac{\partial P}{\partial r}$ is very small, so $(\frac{\partial P}{\partial r})^2$ can be ignored. Furthermore, we can ignore $c(\frac{\partial P}{\partial r})^2$, if $c<1$

Substituting equation (11) in equation (10) and according to the assumptions A and B, we can write equation (10) as follows:

$$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}=\frac{\varphi\mu c}{k}\frac{\partial P}{\partial t} \tag{12}$$

Or:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)=\frac{\varphi\mu c}{k}\frac{\partial P}{\partial t} \tag{13}$$

Assuming c is constant (independent of pressure), $\varphi\mu c/k$ is constant and equations (12) and (13) can be linearized. Now, we define η as:

$$\eta=\frac{k}{\varphi\mu c} \tag{14}$$

Equations (12) and (13) can be written as follows:

$$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}=\frac{1}{\eta}\frac{\partial P}{\partial t} \tag{15}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)=\frac{1}{\eta}\frac{\partial P}{\partial t} \tag{16}$$

This equation is a linear form of the radial diffusion equation and η is known as the diffusion constant [7].

3. Solving the problem in the transient state

In this case, the initial short time is considered and pressure P throughout the reservoir (including $r=r_e$) is a function of time which is not affected by production yet, so $\frac{\partial P}{\partial r}\neq 0$. We consider the followings;

$$P(r,0)=P_0 \qquad r\le r_e \tag{17}$$

$$P(0,t)=P_w \tag{18}$$

$$P(r_w,t)=P_w \tag{19}$$

3.1. Solving the problem by using the separation of variables method

We do the following transform:

$$\Gamma(r,t)=P(r,t)-P_w \tag{20}$$

Hence, we can write equation (15) and conditions (17), (18) and (19) as below:

$$\frac{\partial\Gamma}{\partial t}=\eta\left(\frac{\partial^2\Gamma}{\partial t^2}+\frac{1}{r}\frac{\partial\Gamma}{\partial t}\right) \tag{21}$$

$$\Gamma(r,0)=P_0-P_w=\Gamma_0 \tag{22}$$

$$\Gamma(0,t)=0 \tag{23}$$

$$\Gamma(r_w,t)=0 \tag{24}$$

By using the separation of variables method, we have:

$$\Gamma(r,t)=R(r).Q(t) \tag{25}$$

According to equations (21) and (25):

$$\frac{1}{\eta Q(t)}\frac{dQ}{dt}=\frac{1}{R(r)}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)=-\lambda^2 \tag{26}$$

Note that the separation constant is chosen, the function $R(r)$ will be as orthogonal (Bessel). Therefore, from eqaution (26):

$$\frac{dQ}{dt}+\lambda^2\eta Q=0 \tag{27}$$

and:

$$Q(t)=e^{-\eta\lambda^2 t} \tag{28}$$

We can also have:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{\partial R}{\partial r}+\lambda^2 R=0 \tag{29}$$

Equation (29) is a Bessel equation with the following solution [8]:

$$R(r)=AJ_0(\lambda r)+BY_0(\lambda r) \tag{30}$$

From equation (25) and boundary conditions (23) and (24) we can have:

$$\Gamma(r_w,t)=R(r_w).Q(t)=0 \quad\Rightarrow\quad R(r_w)=0 \tag{31}$$

$$\Gamma(0,t)=R(0).Q(t)=0 \quad\Rightarrow\quad R(0)=0 \tag{32}$$

As second-order Bessel functions are not defined at point zero, therefore in equation (30) when $r=0$ and in according to (32), $B=0$. On the other hand:

$$R(r_w)=AJ_0(\lambda r_w)=0$$

The values of λ are obtained by solving $J_0(\lambda r_w)=0$:

$$\lambda_1 r_w=2.405 \quad\Rightarrow \lambda_1=\frac{2.405}{r_w}$$

$$\lambda_2 r_w=5.520 \quad\Rightarrow \lambda_2=\frac{5.520}{r_w}$$

$$\lambda_3 r_w = 8.654 \quad \Rightarrow \lambda_3 = \frac{8.654}{r_w}$$

$$\lambda_4 r_w = 11.792 \quad \Rightarrow \lambda_4 = \frac{11.792}{r_w}$$

Now, we put the values of $Q(t)$ and $R(t)$ in equation (25):

$$\Gamma(r,t) = \sum_{n=1}^{\infty} A_n e^{-\eta\lambda_n^2 t} J_0(\lambda_n r) \tag{33}$$

By applying the condition (22):

$$\Gamma_0 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \tag{34}$$

Therefore, Γ_0 has been expanded in terms of Bessel sentences. Now, we use the orthogonality property of Bessel functions. We multiply two sides of (34) in $rJ_0(\lambda_n r)dr$ and after integrating, we will have:

$$\int_0^{r_w} \Gamma_0 r J_0(\lambda_n r)dr = A_n \int_0^{r_w} r J_0^2(\lambda_n r)dr \tag{35}$$

And:

$$A_n = \frac{\int_0^{r_w} \Gamma_0 r J_0(\lambda_n r)dr}{\int_0^{r_w} r J_0^2(\lambda_n r)dr} = \frac{\frac{\Gamma_0 r_w}{\lambda_n} J_1(\lambda_n r_w)}{\frac{r_w^2}{2} J_1^2(\lambda_n r_w)} \tag{36}$$

Then:

$$A_n = \frac{2\Gamma_0}{\lambda_n r_w J_1(\lambda_n r_w)} \tag{37}$$

Finally, we write (33) as below:

$$\Gamma(r,t) = \frac{2\Gamma_0}{r_w} \sum_{n=1}^{\infty} e^{-\eta\lambda_n^2 t} \frac{J_0(\lambda_n r)}{(\lambda_n r_w) J_1(\lambda_n r_w)} \tag{38}$$

Or:

$$P(r,t) = P_w + \frac{2(P_0 - P_w)}{r_w} \sum_{n=1}^{\infty} e^{-\eta\lambda_n^2 t} \frac{J_0(\lambda_n r)}{(\lambda_n r_w) J_1(\lambda_n r_w)} \tag{39}$$

3.2. Solving the problem by using Laplace transform

First, we do the following transform:

$$\theta(r,t) = P(r,t) - P_0 \tag{40}$$

We can write equation (15) and conditions (17), (18) and (19) as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta(r,t)}{\partial r}\right) = \frac{1}{\eta}\frac{\partial\theta(r,t)}{\partial t} \tag{41}$$

$$\theta(r,0)=P_0-P_0=0 \quad (42)$$
$$\theta(0,t)=P_w-P_0=\theta_w \quad (43)$$
$$\theta(r_w,t)=P_w-P_0=\theta_w \quad (44)$$

After applying Laplace transform on equation (41), we can have:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{d\theta(r,s)}{dr}\right)=\frac{1}{\eta}(s\theta(r,s)-\theta(r,0)) \quad (45)$$

Now, with regards to equation (42) we can write equation (45) as below:

$$\frac{d^2\theta(r,s)}{dr^2}+\frac{1}{r}\frac{d\theta(r,s)}{dr}-\frac{s}{\eta}\theta(r,s)=0 \quad (46)$$

Or:

$$r^2\theta''+r\theta'-\frac{s}{\eta}r^2\theta=0 \quad (47)$$

Equation (47) is a modified Bessel equation. In general, a modified Bessel equation is presented in the form of equation (48), which has a solution such as equation (49) as bellow [8]:

$$x^2y''+x(a+2bx^m)y'+(c+dx^{2n}-b(1-a-m)x^m+b^2x^{2m}y=0 \quad (48)$$

$$y(x)=x^{\frac{1-a}{2}}e^{-\frac{bx^m}{m}}\left\{AZ_p(\frac{\sqrt{|d|}}{n}x^n)+BZ_{-p}(\frac{\sqrt{|d|}}{n}x^n)\right\} \quad (49)$$

where $p=\frac{1}{n}\sqrt{(\frac{1-a}{2})^2-c}$ and Z is one of the functions I, J, Y, K which are determined by using Table 1.

Table 1. Z in the solution of modified Bessel equation

	Z_{-p}	Z_p	p	$\sqrt{d}$
1	J_{-p}	J_p	Non-integer	Real
2	Y_p	J_p	Integer	Real
3	I_{-p}	I_p	Non-integer	Imaginary
4	K_p	I_p	Integer	Imaginary

Accordingly, the solution for equation (47) will be:

$$\theta(r,s)=AI_0(r\sqrt{\frac{s}{\eta}})+BK_0(r\sqrt{\frac{s}{\eta}}) \quad (50)$$

On the other hand, after applying Laplace transform on (43) and (44):

$$\theta(0,s)=\frac{\theta_w}{s} \quad (51)$$

$$\theta(r_w,s)=\frac{\theta_w}{s} \quad (52)$$

Now, we apply the above conditions on equation (50) and we determine the coefficients *A* and *B*. For $r=0$ the left side of equation (50) is defined but K_0 has an undefined value, hence B must be zero. Now, for $r=r_w$ we can have:

$$\frac{\theta_w}{s}=AI_0(r_w\sqrt{\frac{s}{\eta}})$$

And then:

$$A=\frac{\theta_w}{sI_0(r_w\sqrt{\frac{s}{\eta}})}$$

Accordingly, the solution of the equations (46) and (51) and (52) is:

$$\theta(r,s)=\frac{I_0(r\sqrt{\frac{s}{\eta}})}{sI_0(r_w\sqrt{\frac{s}{\eta}})}\theta_w \tag{53}$$

Now, by applying inverse Laplace transform on the equation above, $\theta(r,t)$ is resulted. There are many techniques to find the inverse Laplace transform of a Laplace domain function [9], but its discussion is outside the scope of this study. On the other hand, by referring to most of tables, any inverse value will not be found for the fraction above, thus, we use Heaviside's theorem as follows.

3.3. Heaviside's expansion theorem [10]

If $P(s)$ is a polynomial with degree less than n and α_k is the roots of $Q(s)=0$, as well as $Q'(\alpha_k)$ is derivative of $Q(s)$ in terms of s at $s=\alpha_k$ then:

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}=\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$$

According to this theorem we can write:

$$L^{-1}\left\{\frac{I_0(r\sqrt{\frac{s}{\eta}})}{sI_0(r_w\sqrt{\frac{s}{\eta}})}\right\}=\sum_{k=0}^{\infty}\frac{P(s_k)}{\left.\frac{dQ}{ds}\right|_{s=s_k}}e^{s_k t} \tag{54}$$

where s_k is the root of the following equation:

$$sI_0(r_w\sqrt{\frac{s}{\eta}})=0$$

In other words:

$$s_0=0$$

$$I_0(r_w\sqrt{\frac{s_k}{\eta}})=0 \qquad k=1,2,3,...$$

There is the following relation between I_n and J_n:

$$I_n(x) = i^{-n} J_n(ix) = e^{\frac{-n\pi i}{2}} J_n(ix) \tag{55}$$

$$J_0(ir_w\sqrt{\frac{s_k}{\eta}}) = 0 \qquad k = 1,2,3,... \tag{56}$$

Or:

$$J_0(\lambda_k) = 0 \qquad s.t \qquad \lambda_k = ir_w\sqrt{\frac{s_k}{\eta}} \qquad k = 1,2,3,... \tag{57}$$

The first four roots of the recent equation are:

$$\lambda_1 = 2.4048 \qquad \Rightarrow s_1 = -\frac{\eta(2.4048)^2}{r_w^2}$$

$$\lambda_2 = 5.5201 \qquad \Rightarrow s_2 = -\frac{\eta(5.5201)^2}{r_w^2}$$

$$\lambda_3 = 8.6537 \qquad \Rightarrow s_3 = -\frac{\eta(8.6537)^2}{r_w^2}$$

$$\lambda_4 = 11.7915 \qquad \Rightarrow s_4 = -\frac{\eta(11.7915)^2}{r_w^2}$$

It can be seen that regarding the values of λ_k, the roots of s_k can be easily obtained.

Now $P(s_k) = I_0(r\sqrt{\frac{s_k}{\eta}})$ must be calculated.

$$P(s_0) = P(0) = I_0(0) = 1$$

$$P(s_k) = I_0(-ir\sqrt{\frac{\lambda_k}{r_w}}) = J_0(\lambda_k \frac{r}{r_w}) \qquad k = 1,2,3,...$$

On the other hand:

$$sI_0(r_w\sqrt{\frac{s}{\eta}}) = 0$$

$$\frac{dQ}{ds} = \frac{d\left(sI_0(r_w\sqrt{\frac{s}{\eta}})\right)}{ds} = I_0(r_w\sqrt{\frac{s}{\eta}}) + \frac{r_w}{2}\sqrt{\frac{s}{\eta}} I_1(r_w\sqrt{\frac{s}{\eta}})$$

$$\left.\frac{dQ}{ds}\right|_{s=0} = I_0(0) = 1$$

$$\left.\frac{dQ}{ds}\right|_{s=s_k} = -\frac{i\lambda_k}{2} I_1(-i\lambda_k) = -\frac{\lambda_k}{2} J_1(\lambda_k) \qquad k = 1,2,3,...$$

Substituting the obtained values in (54) and according to the (53) we can have:

$$\theta(r,t)=\theta_w\left\{1-2\sum_{k=1}^{\infty}\frac{J_0(\lambda_k \frac{r}{r_w})}{\lambda_k J_1(\lambda_k)}e^{\frac{-\eta\lambda_k^2}{r_w^2}t}\right\} \tag{58}$$

Or:

$$P(r,t)=P_0+(P_w-P_0)\left\{1-2\sum_{k=1}^{\infty}\frac{J_0(\lambda_k \frac{r}{r_w})}{\lambda_k J_1(\lambda_k)}e^{\frac{-\eta\lambda_k^2}{r_w^2}t}\right\} \tag{59}$$

4. Conclusion

The recent study has provided a nonlinear radial diffusion equation to achieve the pressure distribution in a petroleum reservoir producing a two-phase hydrocarbon fluid with radial flow. This nonlinear equation does not have any simple solution. In particular conditions, this equation is linearized and solved in the transient state of flow by two analytical methods. In the separation of variables method (based on Bessel equation), P is written in the terms of Bessel sentences. In the second method (based on Laplace transform), we obtained a modified Bessel equation whose solution is determined. Applying the inverse Laplace transform based on Heaviside's theorem, we solved the problem and achieved the pressure distribution which is an applied function in a petroleum reservoir. The assumption of an incompressible fluid such as oil in the problem, is equivalent to assuming that pressure is maintained constant at the well radius ($r=r_w$) and at some external radius. In other words, the entire flow into the well passes across the external radius.

In the continuation of this study, it would be interesting to obtain the distribution of pressure in a reservoir producing the fluid by spherical flow.

Nomenclature

h	*Height of a cylindrical part of reservoir*	$q\rho\vert_r$	*Rate of output mass in cylinder*
dr	*Thickness of a cylindrical part of reservoir*	ρ	*Mass density of fluid*
r	*Inner radius of a cylindrical part of reservoir*	P	*Pressure*
r_e	*Outer diameter of the reservoir*	m	*Mass of fluid*
P_e	*Pressure of the reservoir in radius* r_e	c	*Compressibility of fluid*
r_w	*Radius of well*	μ	*Viscosity of fluid*
q	*Amount of fluid flow*	φ	*Porosity degree of reservoir rock*
$q\rho\vert_{r+dr}$	*Rate of input mass in cylinder*	η	*Diffusion constant*

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To whom correspondence should be addressed: Professor Amir H. Mohammadi, Discipline of Chemical Engineering, School of Engineering, University of KwaZulu-Natal, Howard College Campus, King George V Avenue, Durban 4041, South Africa